Chapter 3

More Proof Techniques

3.1   Proofs by contradiction

The following proof proceeds by contradiction. That is, we will assume that the claim we are trying to prove is wrong and reach a contradiction. If all the derivations along the way are correct, then the only thing that can be wrong is the assumption, which was that the claim we are trying to prove does not hold. This proves that the claim does hold.

Theorem 3.1.1. $\sqrt{2}$ *is irrational.*

Proof. We have seen previously that every integer is either even or odd. That is, for every $n \in \mathbb{Z}$ there exists $k \in \mathbb{Z}$, such that either $n = 2k$ or $n = 2k+1$. Now, if $n = 2k$ then $n^2 = (2k)^2 = 4k^2 = 2 \cdot (2k^2)$, which means that if n is even then n^2 is also even. On the other hand, if $n = 2k+1$ then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1$, so if n is odd then n^2 is also odd.

We now proceed with a proof by contradiction. Assume that $\sqrt{2}$ is rational, that is, $\sqrt{2} \in \mathbb{Q}$. (This is the assumption that should lead to a contradiction.) By definition, this means that there exist two numbers $p, q \in \mathbb{Z}$, with $q \neq 0$, such that

$$\frac{p}{q} = \sqrt{2},$$

and thus

$$\left(\frac{p}{q}\right)^2 = 2.$$

We can assume that p and q have no common divisor, since all common divisors can be divided out to begin with. We have

$$p^2 = 2q^2.$$

This shows that p^2 is even, and consequently p must be even; that is, $p = 2k$ for some $k \in \mathbb{Z}$. Then

$$p^2 = 4k^2 = 2q^2,$$

so
$$2k^2 = q^2.$$

This shows that q^2 is even, and consequently that q is even. Thus both p and q are even, contradicting the fact that p and q have no common divisor. We have reached a contradiction, which completes the proof. $\qquad\square$

Proof Technique: Proof by contradiction. Suppose we want to prove some statement A by contradiction. A common template for such proofs is as follows:

(a) State the method of proof. For example, "The proof proceeds by contradiction."

(b) State the assumption that should lead to the contradiction. For example, "Assume statement A does not hold."

(c) Proceed with a chain of clear statements, each logically following from the previous ones combined with our shared knowledge base. The final statement in the chain should be a contradiction, either of itself (as in, $0 \neq 0$), or of some previous statement in the chain, or of part of our shared knowledge base.

(d) Conclude the proof. For example, "We have reached a contradiction, which completes the proof."

Theorem 3.1.2. $\log_2 3$ *is irrational.*

Proof. The proof proceeds by contradiction. Assume that $\log_2 3$ is rational. By definition, there exist two numbers $p, q \in \mathbb{Z}$, with $q \neq 0$, such that
$$\log_2 3 = \frac{p}{q},$$
which means that
$$2^{\frac{p}{q}} = 3,$$
and thus
$$2^p = 3^q.$$

We can assume that $p, q > 0$. (Indeed, if $p/q > 0$ then we can just work with $|p|$ and $|q|$, and if $p/q \leq 0$ we reach a contradiction of the form $3 = 2^{p/q} \leq 2^0 = 1$.) Now, any positive integer power of 2 is even, because it has 2 as a divisor, so 2^p is even. On the other hand, a positive integer power of 3 is odd, as we've seen previously. We have reached a contradiction. $\qquad\square$

3.2 Direct proofs

We should not forget perhaps the most intuitive proof technique of all: the direct one. Direct proofs start out with our shared knowledge base and, by a sequence of logical derivations, reach the conclusion that needs to be proved. Such proofs are often particularly ingenious and surprising.

Consider the following well-known puzzle question. Take the usual 8×8 chessboard and cut out two diagonally opposite corner squares. Can the remaining 62 squares be tiled by domino-shaped 2×1 tiles, each covering two adjacent squares of the board? (That is, each tile can be placed either horizontally or vertically, so as to precisely cover two squares of the board.)

Theorem 3.2.1. *A tiling as above is impossible.*

Proof. Every tile covers one white square and one black square. Thus in any tiling as above, the number of white squares covered is the same as the number of black ones. The two removed squares have the same color, hence the number of white squares left on the board is not the same as the number of black ones. So the remaining squares cannot be tiled. □

The above proof can also be phrased as a proof by contradiction, or even in terms of induction. However, even though such a phrasing might appear more formal, it is rather unnecessary, as the above proof is already logically sound (which is critical!), and better conveys the power (and dare I say, the beauty) of the argument.

Proof Technique: Direct proof. Here is a common template for direct proofs:

(a) Provide a chain of clear statements, each logically following from our shared knowledge base and the previous ones. The final statement in the chain should be the claim we need to prove.

(b) (Optional.) Conclude the proof. For example, "This completes the proof."